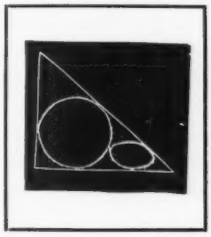
rectification and quadrature of a circle, the new method much more resembling the infinitesimal calculus. He was born in 1739 and died in 1798. He became the head-master of the Seki school. It was one of his most remarkable achievements that the whole perimeter of an ellipse and the length of an elliptic arc were each expressed in an infinite series.

Ajima had written many manuscripts, but there was no printed one. His best disciple Makoto Kusaka (1764-1839) compiled in 1799 the manuscripts into a book which he named Fukyū-Sampō, which means a perpetual mathematics. This however, was not published as intended. The later scholars never failed to copy it and regarded it as one of the most esteemable works. It contains six problems relating to ellipses. The following enunciations will have something of interest, as indicating what problems the old mathematician solved. [The mathematicians of the old school did not enunicate problems minutely, but they drew always such a figure for every problem that we can understand the meaning of the problem; so they usually put the words "as in the figure" in the first part of the enunciation.]



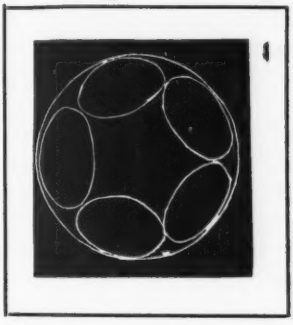
The 6th problem. As in the figure, a circle and an ellipse are inscribed in a right-angled triangle. Given the two sides of the triangle and the minor axes, find the major axes.

The 11th problem. An ellipse and a circle are inscribed in a rectangle, the major axis of the ellipse lying along the diagonal of the rectangle. Given the two sides of the rectangle and the minor axis, find the diameter of the circle.

The 20th problem. A spheroid is cut by a plane. Given the axes of the spheroid and the axes of the section, find the volume of one part of the spheroid.

The 21st problem. A spheroid is inscribed in a rectangular parallelopipedon. Given the three sides of the parallelopipedon and the minor axes of the spheroid, find the major axis.

The 22nd problem. In a given ellipse two equal chords and two equal circles are drawn tangent to the chords and the ellipse. Given the major and minor axes of the ellipse and diameter of the circles, find the length of the chords.



The 23rd problem. As in the figure, a certain number of equal ellipses are inscribed in a circle. Given the diameter of the circle and the major axis and the number of the ellipses, find the minor axis.

Kōhan Sakabe was another of the best pupils of Ajima. He died in 1824, the date of his birth being unknown. In 1810, Sakabe wrote Sampō-Tenzan-Shinan (a guide to the Tenzan method, which resembles the algebra.) One hundred and ninety-six typical problems are arranged in order

from easy problems to difficult ones. At the end of the work, we find the method of finding the length of the whole perimeter and of an arc of an ellipse which had been already discovered by his master Ajima. This was the first appearance of the problems pertaining to ellipses in printed books. It is said by some scholars that the part of the book relating to ellipses had been written by his pupil, Hisanori Kawai.

The successor of Kusaka was Yasushi Wada (or Nei Wada) who died in 1840, the date of birth being unknown. The Enri method was quite reformed by him. His improvement of the method was epoch-making in the history of Japanese mathematics, as Ajima's was. I will next explain his method of finding the perimeter of an ellipse, which is contained in his Enri-shinkō or Elementary lessons on the Enri method.

Divide the major axis $AB=a$ into $2n$ equal parts. Let PQ and $P'Q'$ be the mth and $m+1$th ordinates (called chō), the numbers being given to the ordinates from the minor axis $=b$, taken as the first ordinate to the right-hand ones and to the left-hand ones.

The $(m+1)$th Chōkaku, that is

$$m\text{th chō}-(m+1)\text{th chō}=\left(\frac{m}{n}\right)b \div \sqrt{1-\left(\frac{m}{n}\right)^2}.$$

The $(m+1)$th Haikaku, that is $2\times$elliptic arc intercepted between the mth and $(m+1)$th chō equals

$$\frac{a}{n}\sqrt{1-\left(\frac{m}{n}\right)^2\left(1-\frac{b^2}{a^2}\right)} \div \sqrt{1-\left(\frac{m}{n}\right)^2}.$$

By the binomial expansion,

$$\left[1-\left(\frac{m}{n}\right)^2\left(1-\frac{b^2}{a^2}\right)\right]^{\frac{1}{2}}=1-\frac{1}{2}\left(\frac{m}{n}\right)^2\left(1-\frac{b^2}{a^2}\right)-\frac{1.1}{2.4}\left(\frac{m}{n}\right)^4\left(1-\frac{b^2}{a^2}\right)^2$$

$$-\frac{1.1.3}{2.4.6}\left(\frac{m}{n}\right)^6\left(1-\frac{b^2}{a^2}\right)^4-\frac{1.1.3.5}{2.4.6.8}\left(\frac{m}{n}\right)^8\left(1-\frac{b^2}{a^2}\right)^6-\ldots$$

and

$$\left[1-\left(\frac{m}{n}\right)^2\right]^{-\frac{1}{2}}=1+\frac{1}{2}\left(\frac{m}{n}\right)^2+\frac{1.3}{2.4}\left(\frac{m}{n}\right)^4+\frac{1.3.5}{2.4.6}\left(\frac{m}{n}\right)^6+\ldots$$

By actual multiplication, $\left[1-\left(\frac{m}{n}\right)^2\left(1-\frac{b^2}{a^2}\right)\right]^{\frac{1}{2}}\times\left[1-\left(\frac{m}{n}\right)^2\right]^{-\frac{1}{2}}$ is expanded into a power series of $(m/n)^2$. Then $\frac{1}{2}\times$perimeter of the ellipse$=\lim_{n=\infty}\sum^{m}$ (mth

$$\text{haikaku})=\pi a\left[1-\frac{1}{2}\cdot\frac{1}{2}\left(1-\frac{b^2}{a^2}\right)-\frac{1.1}{2.4}\cdot\frac{1.3}{2.4}\left(1-\frac{b^2}{a^2}\right)^2-\frac{1.1.3}{2.4.6}\cdot\frac{1.3.5}{2.4.6}\left(1-\frac{b^2}{a^2}\right)^3\cdots\right]$$

This was rewritten as follows:

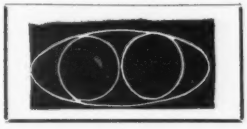
$$\tfrac{1}{2}\times\text{perimeter}=\pi a\left\{1-\frac{1.1}{1^2}\cdot\frac{1-(b^2/a^2)}{4}-\frac{1.1}{1^2}\cdot\frac{1.3}{2^2}\left[\frac{1-(b^2/a^2)}{4}\right]^2-\frac{1.1}{1^2}\cdot\frac{1.3}{2^2}\cdot\frac{3.5}{3^2}\left[\frac{1-(b^2/a^2)}{4}\right]^3-\frac{1.1}{1^2}\cdot\frac{1.3}{2^2}\cdot\frac{3.5}{3^2}\cdot\frac{5.7}{4^2}\left[\frac{1-(b^2/a^2)}{4}\right]^4-\cdots\right\}$$

In this series there appears the factor $1-(b^2/a^2)$, that is the square of eccentricity. This factor was called by the mathematician *ritsu* or modulus of the series.

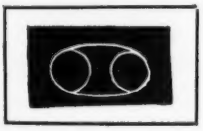
Kyō Uchida, well known by the name of Gokwan Uchida and living up to about 1877, was one of the best pupils of Kusaka and Wada. Kubota, a disciple of Uchida, wrote the first part of Enri-Shōhei-jutsu (the date of publication is not known). In 1855, the second part of the same work was written by Kumamoto, his fellow-student, and revised by Takemura of the same school. The first part is devoted to the methods of finding the center of gravity of plane figures and the second part to those of solid figures. Among the latter, the following problem is found: A spheroid is cut off by a plane parallel to one of its principal planes. Find the center of gravity of the remaining part.

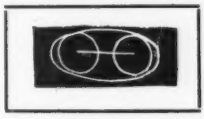
Hiroshi (or Kwan) Hasegawa (1782—1832) was also one of the best pupils of Kusaka, but was expelled from Kusaka's school on account of his bad conduct, and was scorned by the mathematicians of the Seki school. Nevertheless he continued his study. He published many books by the names of his pupils. By the name of Tsunemitsu Murata, he published Sampō-Sokuen-Shōkai in 1831. Fifty problems relating to ellipses are arranged. This work was very famous and was respected by contemporary and later mathematicians. It is the most elegant work devoted to the study of ellipses. It will be interesting for us to know the content of this work, in order to learn the characteristic inclination of study of mathematics peculiar to the old Japanese school.

(1.) As in the figure, two equal circles in external contact are inscribed in an ellipse. Given the major and minor axes, fine the radius of the circles.



(2.) As in the figure, the circles in contact with an ellipse at the extremities of the major axis and having the maximum radius are drawn. Given the major and minor axes of the ellipse, find the maximum radius. [The circles are the osculating circles].



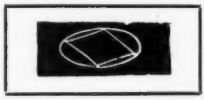


(3.) As in the figure, two equal circles not in contact are in-

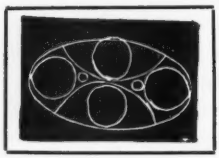
scribed in an ellipse. Given the major and minor axes and the radius of the circles, find the distance of the centers of the two circles.

(4.) Two unequal circles in external contact are inscribed in an ellipse. Given the major and minor axes and the radius of one circle, find the radius of the other circle.

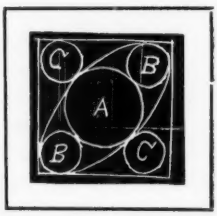
(5.) A rhombus is inscribed in an ellipse. Given the major and minor axes and the area of the rhombus, find the side of the rhombus.



(6.) As in the figure, two equal circular arcs are drawn and four equal large circles and two equal small circles are inscribed within an ellipse. Given the radius of the small circles, find the maximum of the radius of the large circles [two of the large circles are the osculating circles at the extremities of the major axis.]



(7.) As in the figure, an ellipse, one circle A, two equal circles B and two other equal circles C are drawn within a square. Given the radius of the circle C, find the maximum radius of the circle B [the circles B are the osculating circles at the extremities of the major axis.]



(8.) Four equal ellipses and five equal circles are drawn within a square, the extremities of the major axes of two neighboring ellipses being the same point. One of the equal circles has its center at the center of the square to which circle the four ellipses are tangent externally, the other four circles are in the areas formed by the intersection of pairs of ellipses. Given the major axis, find the minor axis.

(9.) Three equal ellipses and one circle are drawn within an equilateral triangle. The major axes of the ellipse coincide with the median of the triangle and the circle is tangent internally to the three ellipses. Given the major and minor axes of the ellipses, find the diameter of the circle.

(10.) As in the figure (23rd problem, p. 173), a certain number *e. g.* five, of equal ellipses are drawn within a circle. Given the diameter of the circle and the number of ellipses, find the major and minor axes.

(11.) As in the figure (23rd problem, p. 173), a certain number *e. g.* five, of equal ellipses are drawn within a circle). Given the major and minor axes, find the number of the ellipses when the diameter of the external circle is minimum.

(12.) Four equal circles are inscribed within the external curvelinear triangular spaces formed by two intersecting equal ellipses [having the same centre and the major axes at right angles]. Given the major axis, find the minor axis when the diameter of the circles is maximum.

(13.) An ellispe and a square are inscribed in a half isosceles trapezoid [the half part of an isosceles trapezoid divided by the join of the middle points of the two parallel sides]. Given the two parallel sides and the distance of the two parallel sides, find the side of the square.

(14.) Two equal chords not intersecting within the ellipse and two unequal circles are drawn within an ellipse, the circles being in internal contact with the ellipse at the extremities of the minor axis and tangent to the chords. Given the major and minor axes and the diameters of both circles, find the length of the chords.

(15.) The enunciation is the same as the preceeding; except that the chords intersect within an ellipse.

(16.) Two equal intersecting chords, two equal ellipses and two unequal circles are drawn within a circle, the circles and ellipses being tangent to the given circle and the intersecting chords. Given the diameter of the external circle and the major and minor axes, find the diameters of the two unequal circles.

(17.) Two circles A and B are intersecting, the center of B lying on the circumference of A; and two equal chords, two equal ellipses tangent to A and the two intersecting chords, and two equal circles C tangent to A, and one small circle D tangent to C and B and the intersecting chords are drawn, the major axes being parallel to the chords respectively. Given the major and minor axes, and the diameter of the circle A, find the diameter of the circle D.

(18.) An ellipse is inscribed in a circle and then a circle B is described on the minor axis of the ellipse. Two equal non-intersecting chords are drawn tangent to B and two equal circles D are inscribed in the smaller segments of the ellipse, cut off by the equal chords. Circles A and C are drawn tangent to the equal chords tangent externally to the ellipse and tangent internally to the given circle. Given the diameters of the given circle and the circle B, find the diameter of the circles D.

(19.) Two equal lines are drawn through the center of a rectangle. In the two pentagonal spaces two equal ellipses are drawn the major axis of which is equal and parallel to the shorter side of the rectangle. In the two triangular spaces, two equal circles A are drawn and two equal circles B are drawn tangent to the ellipses and the two intersecting lines. Given the two sides of the rectangle and the diameter of the circle A, find the diameter of the circle B.

(20.) Two equal ellipses are inscribed in a circle in such a way that three equal pentagons can be inscribed in the segments of the ellipse, a side of each pentagon lying in a common line and one pentagon being common to the two ellipses. Given the diameter of the circle, find the side of the pentagon and the major and minor axes of the ellipses.

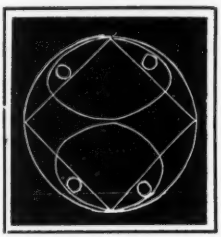
(21.) Two equal pentagons are inscribed in an ellipse in such a way that each of two diameters of the ellipse forms a side each of the two pentagons. Four equal circles are inscribed in the segments cut off by the sides of the pentagons. Given the major axis of the ellipse, find the diameter of the circles.

(22.) Six equal chords, two of which are parallel to the major axis, are drawn within an ellipse, and four equal circles are drawn in the four equal segments. Given the major and minor axes, find the diameter of the circles.

(23.) A chord is drawn in an ellipse and three unequal circles A, B, and

C are drawn, A and B each touching the ellipse in two points and tangent to the chord and C touching the ellipse in one point and tangent to the chord. Given the minor axis and the diameter of the circles A and B, find the diameter of the circle C.

(24.) Five unequal circles are drawn within an ellipse the circles touching the ellipse in two points, and the four chords A, B, C, D touching the two neighboring circles are drawn. Given the three chords A, B, C, find the fourth chord D.



(25.) As in the figure, a square, two equal ellipses and four equal circles are drawn within a circle. Given the diameter of the external circle, find the maximum diameter of the equal circles.

(26.) Two equal circular arcs and four equal small circles are inscribed in the segments of the ellipse cut off by these circular arcs are drawn within an ellipse. Given the major and minor axes, find the diameter of the small circles.

(27.) Two intersecting lines from the vertices of the acute angles and two equal ellipses having the major axes parallel to one side are drawn within a right-angled taiangle. Given the ratio of the major axis to the one side parallel to it, find the ratio of the minor axis to the other side parallel to it.

(28.) Two equal circles A with centers on the major axis, two equal circles B with centers on the minor axis, and one circle C whose center is the center of ellipse and tangent to circles A and B, are drawn within an ellipse. Given the major and minor axes, find the diameter of the circles A.

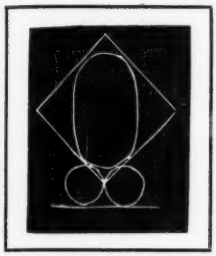
(29.) Circles whose centers are on major axis, and squares one of whose diagonals coincide with major axis are inscribed alternately within an ellipse. Given the major and minor axes, find the number of circles and the number of squares.

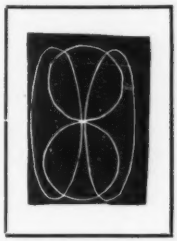
(30) and (31.) Two subsidiary problems of the preceeding.

(32.) An ellipse and a circle A are inscribed in a semi-circle, the major axis of the ellipse being parallel to the base of the semi-circle and the diameter of circle A coinciding with the minor axis of the ellipse. Two equal circles B tangent externally are inscribed in each of the crescents formed by the ellipse and circle A. Given the diameter of the small circles B, find the diameter of the semi-circle when the major axis is a maximum.

(33.) An ellipse is drawn within a square in such a way that it touches three sides of the square and its major axis is parallel to the side of the square. Four equal circles B whose centers form a square whose sides are parallel to the former square are inscribed in an ellipse, also a fifth equal circle B is discribed on the minor axis produced so that it is tangent to the ellipse and the side of the square. Two equal circles A are drawn tangent to this fifth circle B and two sides of the square. Given one side of the square, find the diameter of the circles A.

(34.) As in the figure, two equal circles, an ellipse and a square are drawn above a straight line, the ellipse touching the four sides of the the square and two circles. Given the diameter of the circles and the minor axis, find the major axis.

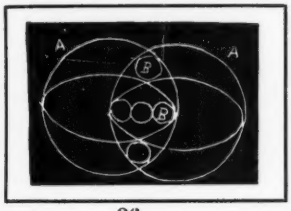




(35.) As in the figure, two equal ellipses and two equal circles are drawn. Given the major and minor axes, find the diameter of the circles.

(36.) As in the figure, two equal circles A and two equal ellipses and five equal small circles B are drawn. Given the diameter of the circles A, find the limiting length of the diameter of the small circles B.

(37.) As in the figure, two equal circles and one ellipse are drawn above a straight line, the major axis being parallel to the straight line. Given the diameter of the circles and the minor axis, find the major axis.



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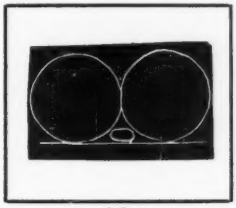

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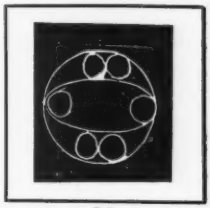

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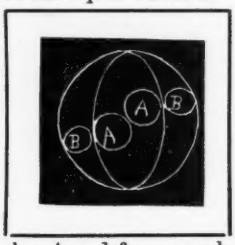
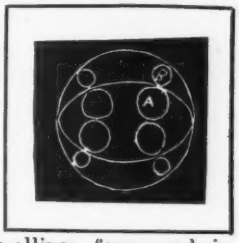
(38.) As in the figure, an ellipse and six equal circles are drawn within a circle. Given the diameter of the external circle, find the maximum diameter of the equal circles.

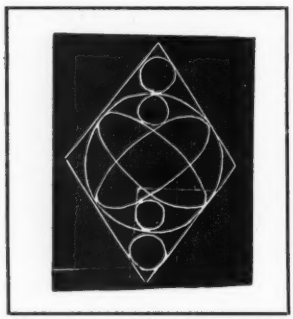


(39.) As in the figure, an ellipse, two equal circles A and two equal circles B are drawn within a circle. Given the diameters of the circles A and B, and the major axis, find the minor axis.



(40). As in the figure, an ellipse, four equal circles A and four equal circles B are drawn within a circle. Given the diameters of the circles A and B, and the major axis, find the minor axis.

(41.) As in the figure, a circle, two equal ellipses, two equal circles A and two equal circles B are drawn within a rhombus. Given the major and minor



axes, and the diameter of the circle A, find the diameter of the circles B.

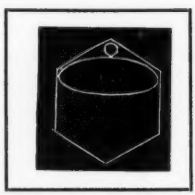
(42.) A spheroid and a certain number of equal small spheres surrounding the spheroid are drawn within a sphere. Given the major and minor axes, and the diameter of the small spheres, find the number of the small spheres.

(43.) Equal ellipses whose major axes coincide with diameters joining opposite vertices of a regular incribed polygon are inscribed within a circle. Equal circles are inscribed in the triangular space formed by two consecutive ellipses and the circle. Given the major and minor axes of the ellipses and the diameter of the small circles, to find the number of ellipses.

(44.) Equal ellipses whose major axes coincide with the diameters joining opposite vertices of a regular inscribed polygon are inscribed in the polygon. In the triangular spaces, formed at the vertices of each ellipse, by three consecutive ellipses, equal circles B are drawn, and in the space common to all the ellipses at the center of the polygon a circle A is drawn. Given the diameter of the external circle, find the diameter of the circle A when the diameter of the circles B is a maximum.

(45.) Two equal spheriods whose major axes are at right angles, and a certain number of equal spheres touching each of the spheroids and the sphere are drawn within a given sphere, the small spheres touching each other in pairs. Given the diameter of the external sphere and the number of the small spheres, find the diameter of the small spheres.

(46.) A circle A is concentric with a given circle. A certain number of equal ellipses tangent to the two circles at the extremities of the major axis are drawn. In the space common to two consecutive ellipses equal circles B are drawn. Given the diameter of the external circle, the major and minor axes, and the number of the ellipses, find the diameter of the circles B.



(47.) As in the figure, an ellipse and a circle are inscribed within a regular polygon. Given the major and minor axes, find the diameter of the circle.

(48.) Equal ellipses are drawn in a regular polygon. The number of ellipses is equal to the number of sides of the polygon; each ellipse is tangent to three consecutive sides of the polygon; and the major axis of each ellipse is parallel to the mid side of the three consecutive sides. In the areas common to two consecutive ellipses equal circles B are drawn. At the center of the polygon a circle A is drawn tangent to the ellipses. Given the side of the polygon, and the diameter of the circle A, find the diameter of the small circles.

(49.) Ellipses are incribed in a polygon as in problem 47. Given the major and minor axes, find the side of the polygon.

(50.) Equal ellipses are drawn within a polygon as in problem 49, and equal circles are drawn in the area common to two consecutive ellipses. Given the major and minor axes, find the diameter of the circles.*

As an appendix of this book, the author Murata described the minute demonstrations of the solutions of the sixth and eleventh problems of Ajima's Fukyū-Sampō in which the solutions of all problems were not demonstrated.

I will conclude this paper by adding that ellipsoids which were treated by Japanese mathematicians of the old school were spheroids only, called Chōryūen or elongated solid circle.

Tokyo, *December 1, 1905.*

ON THE FUNCTIONS WHICH HAVE A GIVEN ALGEBRAICAL ADDITION THEOREM.

By M. KABA, Tokyo, Japan.

Let $f(x)$ denote the uniform analytical function possessing a given algebraical addition theorem, that is,

$$f(x+y)=F[f(x), f(y)]\ldots(1).$$

It is evident that the function satisfies the equation

$$f(2x)=F[f(x), f(x)]\ldots(2).$$

In the first place we prove conversely that the uniform analytical function $\varphi(x)$ which satisfies equation (2) will satisfy equation (1). Set

$$\varphi(x+y)-F[\varphi(x), \varphi(y)]=F_1(x, y).$$

If we use $x+h$ instead of y and develop the function F_1 in a power series of h by Maclaurin's theorem, all the coefficients will be zero. In fact, as the function F is symmetrical in $\varphi(x)$ and $\varphi(y)$, we have the following:

$$\left[\frac{dF}{dh}\right]_{h=0}=\frac{1}{2}\frac{d}{dx}F[\varphi(x), \varphi(x)], \quad \left[\frac{d^2F}{dh^2}\right]_{h=0}=\frac{1}{2^2}\frac{d^2}{dx^2}F[\varphi(x), \varphi(x)], \ldots$$

And also we have the following:

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*In justice to the author and to Editor Dickson, I wish to state that I changed the enunciation of all the problems unaccompanied by diagrams, to avoid making so many wood-cuts. Ed. F.

$$\left[\frac{d\varphi(2x+h)}{dh}\right]_{h=0}=\frac{1}{2}\frac{d\varphi(2x)}{dx},\quad \left[\frac{d^2\varphi(2x+h)}{dh^2}\right]_{h=0}=\frac{1}{2^2}\frac{d^2\varphi(2x)}{dx},\ ..$$

Therefore we have:

$$(F_1)_{h=0}=\varphi(2x)-F[\varphi(x),\ \varphi(x)],$$

$$\left[\frac{dF_1}{dh}\right]_{h=0}=\frac{1}{2}\frac{d}{dx}\Big[\varphi(2x)-F[\varphi(x),\ \varphi(x)]\Big],$$

$$\left[\frac{d^2F_1}{dh^2}\right]_{h=0}=\frac{1}{2^2}\frac{d^2}{dx^2}\Big[\varphi(2x)-F[\varphi(x),\ \varphi(x)]\Big],\ ...$$

But as $\varphi(x)$ satisfies the equation (2), we have:

$$(F_1)_{h=0}=0,\quad \left[\frac{dF_1}{dh}\right]_{h=0}=0,\quad \left[\frac{d^2F_1}{dh^2}\right]_{h=0}=0,\ ...$$

In the second place, we have to prove that there is a single uniform analytical function which satisfies the equation (2).

Let us suppose that the function $f(x)+\theta(x)$ satisfies the equation (2), where $f(x)$ satisfies (1) and therefore satisfies (2). From what we have said above, we know that $f(x)+\theta(x)$ possesses an addition theorem. But $f(x)$ possesses the given addition theorem (1). Therefore $f(x)+\theta(x)$ must be one of the three functions: Algebraical functions, simply periodical functions, doubly periodical functions. Hence $\theta(x)$ also must be one of these three functions, and therefore it possesses an addition theorem; that is

$$\theta(x+y)=F_1[\theta(x),\ \theta(y)],$$

and $\theta(x)$ has satisfied the eqnation

$$\theta(2x)=F_1[\theta(x),\ \theta(x)].$$

By the foregoing hypothesis, we have

$$f(2x)+\theta(2x)=F[f(x)+\theta(x),\ f(x)+\theta(x)],$$

that is,

$$F[f(x),\ f(x)]+F_1[\theta(x),\ \theta(x)]=F[f(x)+\theta(x),\ f(x)+\theta(x)].$$

Let us use y instead of $f(x)$ and z instead of $\theta(x)$, then we shall get the following identity:

$$F_1(z)=F(y+z)-F(y)$$

where F and F_1 are the algebraical functions of y and z. But as the first member of the identity is independent of y,

$$F'(y+z)-F'(y)=0.$$

Then $F'(y)$ must be a constant and $F(y)$ will have the form

$$F(y)=ay+b.$$

Also $f(x)$ and $\theta(x)$ will have the relations, respectively:

$$f(2x)=af(x)+b, \quad \theta(2x)=a\theta(x).$$

Therefore both $f(x)$ and $\theta(x)$ must be constants. Hence we have the following theorem:

The uniform analytical function satisfying a given addition theorem is the only one which satisfies the equation acquired by making $x=y$ on the addition theorem.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

The following note will be of value, since it calls attention to the fact that unless the law of a series is given, the *definite* determination of the series is impossible. If only a certain number of terms of a series is given and not the law of the series, the actual law of the series can in no case be really determined. All that can be done is to find the *simplest law* which the few given terms will obey. The solutions referred to may be justified on the ground that they were attempts to find this *simplest law*. ED. F.

NOTE ON THE SOLUTION OF PROBLEM 266, IN AUGUST-SEPTEMBER MONTHLY.

By CLARENCE E. COMSTOCK, Bradley Polytechnic Institute.

Find the nth term and the sum of n terms of the series $1+3+7+17+\ldots$

One writer assumes without the least excuse for such an assumption that it is an arithmetical series of the third order and proceeds by the method of finite differences, getting a result in accord with his assumption. Five terms of his series are $1+3+7+17+37+\ldots$

Another writer assumes with a little more plausible excuse that it is a recurring series of the second order and, of course, gets a result in accord with his assumption. Five terms of his series are $1+3+7+17+41+\ldots$

Suppose I make the assumption that it is an arithmetical series of the fourth order. I can then build up

$$1+3+7+17+38+76+\ldots \text{ or } 1+3+7+17+39+81+\ldots$$

or as many more as I care to take the time to construct, on the one supposition that it is an arithmetical series of the fourth order. Or let me assume it a recurring series of the fourth order, and get, using the scale of relation,

$$u_n=2u_{n-1}+u_{n-2}+0u_{n-3}+u_{n-4}, \text{ say, } 1+3+7+17+42+104+\ldots$$

The fact is, the problem is absolutely without meaning. There is nothing in the sequence $1+3+7+17+\ldots$ to show what the next term is. A series is not determined until its law is in some way stated or indicated.

267 Proposed by O. E. GLENN, Ph. D., Philadelphia, Pa.

Express the trigonometric functions of x as infinite continued fractions.

Solution by R. D. CARMICHAEL, Professor of Mathematics, Anniston, Ala.; and J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

Let the series $u=A-B+C-D\ldots=\dfrac{a_1}{c_1}+\dfrac{a_2}{c_2}+\dfrac{a_3}{c_3}+\ldots$

To determine the a's we have the following (Euler's "Introductio," I, §367):

$$a_1=A.c_1,\ a_2=\frac{B.c_1c_2}{A-B},\ a_3=\frac{AC.c_2c_3}{(A-B)\,(B-C)},\ a_4=\frac{BD.c_3c_4}{(B-C)\,(C-D)}\cdots$$

in which the c's are to be so chosen that the a's are integral functions. Applying these formulas, we have

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots=\frac{x}{1}+\frac{x^2}{2.3-x^2}+\frac{2.3x^2}{4.5-x^2}+\ldots$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots=\frac{1}{1}+\frac{x^2}{1.2-x^2}+\frac{1.2x^2}{3.4-x^2}+\frac{3.4x^2}{5.6-x^2}+\ldots$$

$$\tan x=\frac{2}{\pi-2x}-\frac{2}{\pi+2x}+\frac{2}{3\,\pi-2x}-\frac{2}{3\,\pi+2x}+\frac{2}{5\,\pi-2x}+\ldots*$$

$$=\frac{2}{\pi-2x}+\frac{4x}{(\pi-2x)^2}+\frac{2\pi}{\pi^4-4x^2}+\frac{(\pi-2x)(3\pi-2x)^2}{2x(\pi+2x)}+\ldots$$

Encyclopedia Britannica, Vol. XXIII, p. 572.

$$\cot x = \frac{1}{x} - \frac{1}{\pi - x} + \frac{1}{\pi + x} - \frac{1}{2\pi - x} + \frac{1}{2\pi + x} - \ldots *$$

$$= \frac{1}{x} + \frac{x^2}{\pi - 2x} + \frac{(\pi - x)^2}{2x} + \frac{(\pi + x)^2}{(\pi - 2x)} + \ldots$$

$$\sec x = \frac{4\pi}{\pi^2 - 4x^2} - \frac{3.4\pi}{3^2\pi^2 - 4x^2} + \frac{4.5\pi}{5^2\pi^2 - 4x^2} - \frac{4.7\pi}{7^2\pi^2 - 4x^2} + \frac{4.9\pi}{9^2\pi^2 - 4x^2} - \ldots \dagger$$

$$= \frac{4\pi}{\pi^2 - 4x^2} + \frac{3(\pi^2 - 4x^2)^2}{6\pi^2 + 8x^2} + \frac{(9\pi^2 - 4x^2)^2}{30\pi^2 + 8x^2} + \ldots$$

$$\csc x = \frac{1}{x} + \frac{2x}{\pi^2 - x^2} - \frac{2x}{2^2\pi^2 - x^2} + \frac{2x}{3^2\pi^2 - x^2} - \frac{2x}{4^2\pi^2 - x^2} + \ldots$$

$$= a - \frac{ax - 1}{x} + \frac{2x}{\pi^2 - x^2} - \frac{2x}{4\pi^2 - x^2} + \ldots \ddagger$$

$$= \frac{a}{1} + \frac{ax - 1}{1} + \frac{2ax^3}{a\pi^2 x - \pi^2 - x^2 - ax^3} + \frac{2x(ax - 2)(\pi^2 - x^2)^2}{6\pi^2} + \ldots$$

Also solved by G. W. Greenwood.

268. Proposed by O. E. GLENN, Ph. D., Philadelphia, Pa.

Express the hyperbolic functions of x in the form of infinite continued fractions.

Solution by J. SCHEFFER, A. M., Professor of Mathematics, Kee Mar College, Hagerstown, Md.

For the conversion of $\tan x$ and $\tanh x$, Legendre has given a very elegant solution.

Putting $f(x) = 1 + \frac{a}{x} + \frac{1}{2} \cdot \frac{a^2}{x(x+1)} + \frac{1}{2.3} \cdot \frac{a^3}{x(x+1)(x+2)} + \ldots$ we have

$$f(x+1) = 1 + \frac{a}{x+1} + \frac{1}{2} \cdot \frac{a^2}{(x+1)(x+2)} + \frac{1}{2.3} \cdot \frac{a^3}{(x+1)(x+2)(x+3)} + \ldots$$

$$\therefore f(x) - f(x+1) = \frac{a}{x(x+1)} + \frac{a^2}{x(x+1)(x+2)} + \frac{1}{2} \frac{a^3}{x(x+1)(x+2)(x+3)} + \ldots$$

**Encyclopedia Britannica*, Vol. XXIII, p. 572.

†Euler's *Institu. Calc. Diff.*, II, 8, No. 223—224.

‡The quantity a may have any integral value which will avoid negative terms in the continued fraction.

$$=\frac{a}{x(x+1)}\left[1+\frac{a}{x+2}+\tfrac{1}{2}\frac{a^2}{(x+2)(x+3)}+\dots\right]=\frac{a}{x(x+1)}f(x+2);$$

whence, putting $F(x)=\frac{a}{x}.\frac{f(x+1)}{f(x)}$, $F(x+1)=\frac{a}{x+1}\frac{f(x+2)}{f(x+1)}$, we obtain

$$\frac{x}{a}\frac{f(x)}{f(x+1)}-\frac{x}{a}=\frac{1}{x+1}.\frac{f(x+2)}{f(x+1)}.\quad \frac{1}{F(x)}-\frac{x}{a}=\frac{F(x+1)}{a},\quad F(x)=\frac{a}{x+F(x+1)}$$

$$\therefore F(x+1)=\frac{a}{x+1+F(x+1)},\ F(x+2)=\frac{a}{x+2+F(x+2)},\ \text{etc.}$$

$$\therefore F(x)=\frac{a}{x}+\frac{a}{x+1}+\frac{a}{x+2}+\frac{a}{x+3}+\dots,$$

$$\text{where } F(x)=\frac{a}{x}\frac{1+\frac{a}{x+1}+\frac{1}{2}\frac{a^2}{(x+1)(x+2)}+\frac{1}{2.3}\frac{a^3}{(x+1)(x+2)(x+3)}+\dots}{1+\frac{a}{x}+\frac{1}{2}\frac{a^2}{x(x+1)}+\frac{1}{2.3}\frac{a^3}{x(x+1)(x+2)}+\dots}$$

$$\text{For } x=\tfrac{1}{2},\ F(x)=2a.\frac{1+\frac{4a}{2.3}+\frac{16a^2}{2.3.4.5}+\frac{64a^3}{2.3.4.5.6.7}+\dots}{1+\frac{4a}{2}+\frac{16a^2}{2.3.4}+\frac{64a^3}{2.3.4.5.6}+\dots}=\frac{e^{2\sqrt{a}}-e^{-2\sqrt{a}}}{e^{2\sqrt{a}}+e^{-2\sqrt{a}}}\sqrt{a}.$$

$$\therefore \frac{e^{2\sqrt{a}}-e^{-2\sqrt{a}}}{e^{2\sqrt{a}}+e^{-2\sqrt{a}}}2\sqrt{a}=\frac{4a}{1}+\frac{4a}{4}+\frac{4a}{5}+\dots$$

$$\text{Putting } 2\sqrt{a}=x,\ \frac{e^x-e^{-x}}{e^x+e^{-x}}=\tanh x=\frac{x}{1}+\frac{x^2}{3}+\frac{x^2}{5}+\frac{x^3}{7}+\dots$$

$$\text{Also } \frac{e^{ix}-e^{-ix}}{e^{ix}+e^{-ix}}=\frac{ix}{1}-\frac{x^2}{3}-\frac{x^2}{5}-\dots$$

$$\therefore i\tan x=\frac{ix}{1}-\frac{x^2}{3}-\frac{x^2}{5}-\frac{ix^3}{7}-\dots \qquad \text{or, } \tan x=\frac{x}{1}-\frac{x^2}{3}-\frac{x^2}{5}-\dots$$

a very simple and remarkable continued fraction.

Since $\sinh x = -i\sin ix$, and $\cosh x = \cos ix$, we get, by substituting ix for x in the continued fractions for $\sin x$ and $\cos x$,

$$\sinh x = \frac{x}{1} - \frac{x^2}{2.3+x^2} - \frac{2.3x^2}{4.5+x^2} - \frac{4.5x^2}{6.7+x^2} - \text{ etc.},$$

$$\cosh x = \frac{1}{1} - \frac{1}{2+x^2} - \frac{2x^2}{3.4+x^2} - \frac{3.4x^2}{5.6+x^2} - \text{ etc.}$$

GEOMETRY.

289 (Incorrectly numbered 288). Proposed by C. N. SCHMALL, College of the City of New York.

From a point P on a given circle to draw two chords such that, (α) chord PA : chord $PB = m : n$ (a given ratio), and, (β) arc PA : arc $PB = 1 : 3$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let O be the center of the circle, radius r. Also let angle $POA = 2\theta$, angle $POB = 6\theta$. Then $PA = 2r\sin\theta$, $PB = 2r\sin 3\theta$.

$$\frac{PA}{PB} = \frac{\sin\theta}{\sin 3\theta} = \frac{1}{3-4\sin^2\theta} = \frac{m}{n}. \quad \therefore \sin\theta = \tfrac{1}{2}\sqrt{\frac{3m-n}{m}},\ \sin 3\theta = \frac{n}{2m}\sqrt{\frac{3m-n}{m}}.$$

$\therefore$ Make angle $POA = 2\sin^{-1}\frac{1}{2}\sqrt{\frac{3m-n}{m}}$; angle $POB = 2\sin^{-1}\frac{n}{2m}\sqrt{\frac{3m-n}{m}}$.

Then chord PA : chord $PB = m : n$; arc PA : arc $PB = 1 : 3$.

290 (Incorrectly numbered 289). Proposed by J. J. QUINN, Ph. D., Scottdale, Pa.

(a) Suppose a circle described around the origin. Then at the end of a uniformly revolving radius r, a line equal to the diameter is pivoted. Find the equation of the locus of its extremity, if for every unit of angle its projection on the X axis is a constant linear unit, being the same part of the diameter as the angle is of π radians.

(b) Show how it can be applied to the trisection or multisection of an angle.

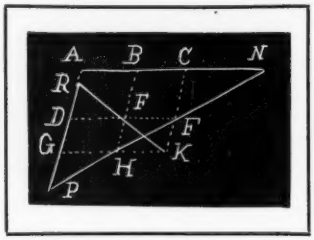
No solution has been received.

292 (Incorrectly numbered 290). Proposed by DR. L. E. DICKSON, The University of Chicago, Chicago, Ill.

Given nine points lying by threes in three columns and in three rows, draw through them, by continuous motion, a broken line composed of only four straight segments, and passing but once through each of the nine points. [A current puzzle.]

Solution by G. I. HOPKINS, A. M., Professor of Mathematics and Astronomy, Manchester (N. H.) High School, and MISS IDA M. SCHOTTENFELS, A. M., New York, N. Y.

Let A, B, C, D, E, F, G, H, and K be the points. It is evident that A, C, K, and G will be the vertices of a parallelogram. Let BH be a median of this parallelogram, and E any point in the median except the center. Then the broken line $ANPRK$ will fulfill the conditions of the problem. This course fails if the middle row and middle column bisect each other. If the row DEF is not parallel to GK, then three lines, or a broken line of three segments will fulfill the conditions of the problem.



G. I. HOPKINS.

If BH and DF are not medians, take the course KER, RP, PN, NB. If BH and DF are medians, take the course KEA, AN, NP, PD; or KEA, AP, PN, NB.

IDA M. SCHOTTENFELS.

Also solved by the Proposer.

296 (Incorrectly numbered 294). Proposed by JOHN JAMES QUINN, Ph. D., Scottdale, Pa.

a) Suppose an indefinite line be pivoted at the end of a revolving radius whose center is the origin; and the initial position of the radius is coincident with the X-axis and the pivoted line perpendicular to it. As the radius revolves through equal amounts of arc the line moves to the right over corresponding equal intercepts on the X-axis. What is the equation of the locus of a point on the line whose distance from the end of the radius is equal to a diameter?

b) Show how the locus can be applied to the multisection of an angle.

c) Suppose the diameter be laid off in both directions.

No solution of this problem has been received.

297 (Incorrectly numbered 295). Proposed by S. F. NORRIS, Professor of Mathematics, Baltimore City College, Md.

One side and the opposite angle of a triangle are fixed. Find the locus of the center of the inscribed circle. Solve by methods of analytic geometry.

I. Solution by C. N. SCHMALL, A. B., 89 Columbia Street, New York City.

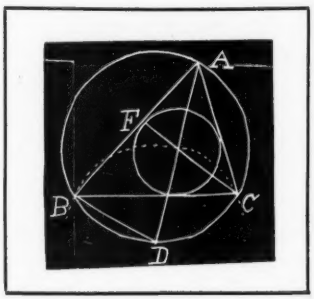
This problem can be solved more easily and more neatly by Euclidean Geometry. Thus, referring to figure, we have,

$$\angle DCO = \angle DCB + \angle BCO = \angle DCB + \tfrac{1}{2}\angle BCA,$$
$$\angle DOC = \angle AOF = \angle OAC + \angle OCA;$$

but $\angle OAC = \angle DCB$ (since arc $DC = DB$), and $\angle OCA = \frac{1}{2}\angle BCA$. Hence $\angle DCO = \angle DOC$.

Therefore $DC = DO$.

Hence, keeping BC constant and vertex A always on the arc BAC (making opposite angle constant) the locus of center O of the inscribed circle is a circle whose center is D and radius DC.



II. Solution by G. W. GREENWOOD, A. M., Dunbar, Pa.

Taking as x-axis the fixed side AB $(=2a)$ and its mid-point as origin, the in-center, since it lies on the bisectors of the angles A and B, must satisfy the equations $y=m_1(x-a)$, $y=m_2(x+a)$, where

$$m_1=\tan\left(\pi-\frac{A}{2}\right)=-\tan\frac{A}{2},\quad m_2=\tan\frac{B}{2}.$$

The opposite angle being constant, $A+B$ is constant. Hence

$$\tan\left(\frac{A}{2}+\frac{B}{2}\right)=\frac{-m_1+m_2}{1+m_1m_2}=\text{a constant}=c, \text{ say.}$$

Hence the in-center satisfies the equation $c(x^2+y^2-a^2)+2ay=0$.

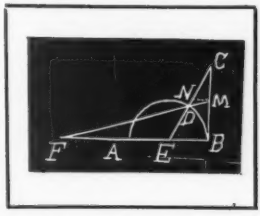
Also solved by J. Scheffer, William Hoover, and L. E. Newcomb.

298 (Incorrectly numbered 296). Proposed by J. J. QUINN, Ph. D., Scottdale, Pa.

Given $AB=BC$ perpendicular to each other, and E and M their mid-points, respectively. On AB describe a semi-circle, and draw CE to meet the circumference in D. Draw DM cutting AB in F. In what ratio is AB divided by the point F?

Solution by C. N. SCHMALL, A. B., 89 Columbia Street, New York; L. E. NEWCOMB, Los Gatos, California; and A. H. HOLMES, Brunswick, Maine.

From the figure, constructed as described in the problem, we have



$FE : ED=NM : ND$. But $NM=\frac{1}{2}EB$, and

$$ND=CD-CN=CD-\tfrac{1}{2}CE\ldots(1).$$
$$\therefore FE : ED=\tfrac{1}{2}EB : CD-\tfrac{1}{2}CE\ldots(2).$$

Also $AB^2=CD.CD'$, where D' is the second intersection of the secant, CE, with the circle. Now let $FA=x$, $EB=r$, $CD=a$. Then $FE=x+r$, $CE=a+r$. Substituting these values in (1) and (2), we have

$$x+r : r=\tfrac{1}{2}r : a-\tfrac{1}{2}(a+r)\ldots(3),$$
$$(2r)^2=a(a+2r)\ldots(4).$$

From (4), $a=r(\pm\sqrt{5}-1)$, and from (3),

$$a=r+\frac{r^2}{x+r}.\quad\therefore r(\pm\sqrt{4}-1)=r+\frac{r^2}{x+r}.$$

Whence $x=(\pm\sqrt{5}+1)r$. $\therefore FA/AB=\frac{1}{2}(\pm\sqrt{5}+1)$.

Also solved by G. W. Greenwood and J. Scheffer.

220 (Incorrectly numbered 219). For solution, see page 85.

221 (Incorrectly numbered 220, p. 90. For solution see page 148.

222 (Incorrectly numbered 221, p. 117). Proposed by Professor F. ANDEREGG, Oberlin College, Oberlin, O.

If a, b, c, ... represent all the prime numbers 2, 3, 5, ... prove that

$$(1+\frac{1}{a^2})(1+\frac{1}{b^2})(1+\frac{1}{c^2})\ldots\ldots=\frac{15}{\pi^2}.$$

No solution has been received.

223 (Incorrectly numbered 222, page 117). Proposed by REV. R. D. CARMICHAEL, Anniston, Ala.

Evaluate $\int_0^1 (1+x^m)^n \log x\, dx$.

Solution by G. W. GREENWOOD, M. A., Dunbar, Pa.

Assuming that n is integral, we have

$$\int_0^1 (1+x^m)^n \log x\, dx = \int \sum_{r=0}^{n} c_r x^{mr} \log x\, dx = \sum \frac{c_r[(mr+1)x^{mr+1}\log x - x^{mr+1}]}{(mr+1)^2}\Big]_0^1$$

$$= -\sum \frac{c_r}{(mr+1)^2}, \text{ since } \lim_{x \doteq 0} x^n \log x = 0.$$

224 (Incorrectly numbered 221, page 153). Proposed by REV. R. D. CARMICHAEL, Anniston, Ala.

Find $\lim_{x \doteq 0} \tan^{-1}x(\log x)$.

Solution by EDWIN L. RICH, Schenectady, N. Y.

$$\lim_{x \doteq 0} \tan^{-1}x \log x = \left.\frac{\log x}{\frac{1}{\tan^{-1}x}}\right\}_{x \doteq 0} = \left.\frac{\frac{d}{dx}(\log x)}{\frac{d}{dx}\left(\frac{1}{\tan^{-1}x}\right)}\right\}_{x \doteq 0}$$

$$= \frac{x^2+1}{x}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \text{ etc.}\right)^2\Big]_{x \doteq 0}$$

$$= (x^2+1)(1-\tfrac{1}{3}x^2+\tfrac{1}{5}x^4-\tfrac{1}{7}x^6+\text{ etc.})(x-\tfrac{1}{3}x^3+\tfrac{1}{5}x^5-\tfrac{1}{7}x^7+\text{ etc.})]_{x \doteq 0} = 0.$$

Also solved by J. Scheffer and G. W. Greenwood.

225 (Incorrectly numbered 222). Proposed by REV. R. D. CARMICHAEL, Anniston, Ala.

If $s_n = 2\left(\frac{1}{n} - \frac{2}{2n^3} + \frac{1}{5n^5} + \frac{1}{7n^7} - \frac{2}{9n^9} + \frac{1}{11n^{11}} + \ldots\right)$ prove that

$$\log 3 = s_3 + s_4,$$
$$\log 7 = s_2 + s_3 + s_4,$$
$$\log 13 = s_2 + 2s_3 + s_4.$$

Solution by J. SCHEFFER, A. M., Professor of Mathematics, Kee Mar College, Hagerstown, Md.

We have $\log(1+\frac{1}{x}) = \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \frac{1}{4x^4} + \ldots$, and

$$\log(1-\frac{1}{x}) = -\frac{1}{x} - \frac{1}{2x^2} - \frac{1}{3x^3} - \frac{1}{4x^4} - \ldots$$

$$\therefore \tfrac{1}{2}\log\left(\frac{1+(1/x)}{1-(1/x)}\right) = \tfrac{1}{2}\log\left(\frac{x+1}{x-1}\right) = \frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \frac{1}{7x^7} + \frac{1}{9x^9} + \ldots$$

Likewise, $\frac{1}{2}\log\left(\frac{x^3+1}{x^3-1}\right) = \frac{1}{x^3} + \frac{1}{3x^9} + \frac{1}{5x^{15}} + \ldots$

$$\therefore \tfrac{1}{2}\log\left(\frac{x+1}{x-1}\right) - \tfrac{1}{2}\log\left(\frac{x^3+1}{x^3-1}\right) = \tfrac{1}{2}\log\left(\frac{x^2+x+1}{x^2-x+1}\right) = \frac{1}{x} - \frac{2}{3x^3} + \frac{1}{5x^5} + \frac{1}{7x^7}$$

$$- \frac{1}{9x^9} + \ldots \text{ or } s_n = \log\left(\frac{n^2+n+1}{n^2-n+1}\right).$$

$\therefore s_3 = \log\frac{13}{7}$ and $s_4 = \log\frac{21}{13}$. $\therefore s_3 + s_4 = \log(\frac{13}{7} \times \frac{21}{13}) = \log 3$. Likewise,

$s_2 + s_3 + s_4 = \log\frac{13}{3} + \log\frac{13}{7} + \log\frac{21}{13} = \log(\frac{13}{3} \cdot \frac{13}{7} \cdot \frac{21}{13}) = \log 13$, and

$$s_2 + 2s_3 + s_4 = \log\tfrac{7}{3} + 2\log\tfrac{13}{7} + \log\tfrac{21}{13} = \log(\tfrac{7}{3} \cdot \tfrac{169}{49} \cdot \tfrac{21}{13}) = \log 13.$$

DIOPHANTINE ANALYSIS.

137 (Incorrectly numbered 136). Proposed by A. H. HOLMES, Brunswick, Maine.

Given $7x^2 - 111 = y^2$. Required a value for y greater than unity which shall be a prime integer.

II. Solution by the PROPOSER.

$7x^2 - 111 = y^2$. Put $x = u + a$. Then $7u^2 + 14au + 7a^2 - 111 = y^2 = [pu + \sqrt{(7a^2 - 111)}]^2$; $7u^2 + 14au = p^2u^2 + 2pu\sqrt{(7a^2 - 111)}$.

$$\therefore u = \frac{14a - 2p\sqrt{(7a^2 - 111)}}{P^2 - 7}.$$

Put $p = 3$. $\therefore u = 7a - 3\sqrt{(7a^2 - 111)}$.

Put $a = 5$. Then $u = 35 - 24 = 11$, and $x = 16$.

$7 \times 16^2 - 111 = 41^2$. $\therefore y = 41$.

This solution shows that Professor Scheffer's solution in the June-July number, p. 148, did not yield the *least* prime, satisfying the equation, as is there stated. Ed. F.

AVERAGE AND PROBABILITY.

167. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

A line l is divided into n segments by $n-1$ points taken at random on it; find the mean value of the product of p of the segments, the p segments being taken at random and p being less than n.

Solution by the PROPOSER.

Let the p segments taken at random in one instance be $a, b, c, \ldots$ If p other points are taken at random on the line, the chance that the first will fall on a is a/l; that the second will fall on b is b/l; and so on. Hence, the chance that the p points will all fall on $a, b, c, \ldots$ in a given order is $M(a.b.c\ldots)/l^p$; and the chance that they will so fall in any order is $(1.2.3\ldots p)M(a.b.c\ldots)/l^p$.

The whole number of ways in which the $n+p-1$ points can be arranged is $(n+p-1)!$ The whole number of ways in which p points can fall on p segments (each in any order) is $(p!)(p!)$. We may then have

$$\frac{(1.2.3\ldots p)\,M(a.b.c\ldots)}{l^p}=\frac{(1.2.3\ldots p)^2}{1.2.3\ldots(n+p-1)}.$$

Hence, $M(a.b.c\ldots)=\dfrac{(1.2.3\ldots p)l^p}{1.2.3\ldots(n+p-1)}$.

172. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A circular arc, with center at one corner of a given square, is drawn through a point taken at random in the square. What is the average length of the arc within the square?

Solution by B. F. FINKEL, Ph. D., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Let $ABCD$ be the square, whose side is a, and let the coördinates of the random point P be (x, y), A being origin, AB the x axis and AD the y axis. Then the length of the arc through the random point is $2\phi\sqrt{(x^2+y^2)}$, where 2ϕ is the angle between the lines drawn from the origin to the intersection of the arc with the sides of the square. Passing to polar coördinates, we have $2\phi\rho$ for the length of the arc. But $\phi=\frac{1}{4}\pi-\sec^{-1}(\rho/a)$. Hence, the length of the arc is $2\rho[\frac{1}{4}\pi-\sec^{-1}(\rho/a)]=f(\rho)$, (say). Then the average length of the arc is

$$\Delta=\frac{f(\rho_1)+f(\rho_2)+\ldots+f(\rho_n)}{n}, \text{ where } n \text{ is the number of arcs,}$$

$$=\frac{\displaystyle\int_0^a 2\rho\frac{\pi}{4}d\rho+2\int_a^{\sqrt{2}a}\rho\left(\frac{\pi}{4}-\sec^{-1}\frac{\rho}{a}\right)d\rho}{\displaystyle\int_0^a d\rho+\int_a^{\sqrt{2}a}d\rho}$$

$$=\left[2\int_0^{\sqrt{2}a}\rho.\frac{\pi}{4}.d\rho-2\int_a^{\sqrt{2}a}\left(\sec^{-1}\frac{\rho}{a}\right)d\rho\right]/\sqrt{2}\,a=\tfrac{1}{2}\sqrt{2}a.$$

Solved in a similar manner with the same result by G. B. M. Zerr.
Solved in an entirely different manner with the result

$$\Delta=4a/3[\sqrt{2}-1-\int_0^{\frac{1}{4}\pi}\log(\tan\tfrac{1}{4}\pi+\tfrac{1}{2}\theta)d\theta]$$

by Henry Heaton. Mr. Heaton assumes that the number of arcs of any given length within the square is proportional to the lengths of the arcs, and the whole number of arcs is equal to the number of points in the square. Now there is no reason why such an assumption may not be made, but such an assumption is certainly highly artificial. In our solution, it is clear that for every point we get a corresponding arc, and but one. If we take in all possible points in the square, we get all possible arcs. How to get all possible points is an open question, and an indefinite number of assumptions may be made as regards the distribution of the points when no law of distribution is given in the problem. The above solution tacitly assumes, (1) That the random points are distributed at equal angular distances on the arcs of circles, and (2) that the arcs of the circles cut the diagonal of the square at equal distances apart. By these assumptions every point in the square is considered. But either or both of these assumptions may be changed in any way at pleasure, each change giving different answers. Ed. F.

PROBLEMS FOR SOLUTION.

ALGEBRA.

273. Proposed by THEODORE L. DELAND, Treasury Department, Washington, D. C.

Three ingots of the precious metals were received at the Mint for assay, where it was found as follows: That in 3 grains of the first ingot and 2 grains of the second the gold was 3 times the silver; that in 2 grains of the first and 6 grains of the third the gold was 8 times the copper; that in 2 grains of the second and 3 grains of the third the silver was 5 times the copper; that in 1 grain of the first, 2 grains of the second, and 3 grains of the third the gold was 2 times the silver; that in 1 grain each of the first and second ingots there were 11 parts of gold to 5 parts of silver; and that 6 grains of the first, 5 grains of the second, and 2 grains of the third on being assayed proved to be 17 carats gold fine. There was no trace of any other metal in the ingots.

Required: The theoretical analysis of each of the three ingots.

274. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Find the limit of $\frac{3^2+1}{3^2-1}\cdot\frac{5^2+1}{5^2-1}\cdot\frac{7^2+1}{7^2-1}\cdot\frac{11^2+1}{11^2-1}\cdots$ where the squared numbers are the natural odd *primes* in order.

GEOMETRY.

303. Proposed by FRANCIS RUST, C. E., Allegheny, Pa.

Prove that the pedal line of any point on a triangle's circum-circle bisects the distance from this point to the triangle's ortho-center.

304. Proposed by G. W. GREENWOOD, M. A., Dunbar, Pa.

Find the tangent at the points $(a, 0)$ and $(0, a)$ to the locus $x^3+y^3=a^3$, and show that these points are points of inflection.

305. Proposed by J. J. QUINN, Ph. D., Scottdale, Pa.

1. Suppose two radii R and R_1 revolve uniformly in the ratio 2 : 3. Find the equation of the locus of the intersection of R with the chord drawn from the end of the diameter to the extremity of R_1. 2. If the chord be drawn to the end of the diameter to the extremity of R, what is the locus of the intersection with R_1? 3. Show how an angle can be trisected by means of this curve.

CALCULUS.

230. Proposed by C. N. SCHMALL, College of the City of New York.

The greatest rectangle is inscribed in an ellipse, and the greatest ellipse in that rectangle, again the greatest rectangle in that (second) ellipse, and the greatest ellipse in that (second) rectangle, and so on *ad infinitum;* show that the sum of all the inscribed rectangles is equal to the area of the rectangle circumscribed about the given ellipse.

231. Proposed by EVA S. MAGLOTT, A. M., Professor of Mathematics, Ohio Northern University, Ada, O.

If a right circular cone stands on an ellipse, prove that the convex surface of the cone is $\frac{1}{2}\pi(OA+OA')(OA.OA')^{\frac{1}{2}}\sin\alpha$, where O is the vertex of ths cone, A and A' the extremities of the major axis of the ellipse, and α is the semi-angle of the cone at the vertex, using the formula $ds=\frac{1}{2}\rho\sqrt{(\rho^2+p^2)}d\theta$, where p is the perpendicular from the vertex to the base of the cone, ρ the distance from the foot of the perpendicular to any point in the perimeter of the base, and θ the angle between the major axis and ρ.

MECHANICS.

194. Proposed by W. J. GREENSTREET, M. A., Editor of the Mathematical Gazette, Stroud, England.

A body has a plane face resting on a rough wedge. The wedge is on a rough inclined plane, thick end down and thin edge horizontal. Find the condition that the body will slide down the wedge with constant acceleration, the wedge not slipping the while. Discuss the case in which the angle of friction for wedge and plane is greater than the angle of inclination of the plane.

195. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

Particles slide from rest at the focus of a parabola, whose axis is vertical, down radius vectors, and are then allowed to move freely. Find the locus of the foci of their subsequent paths.

DIOPHANTINE ANALYSIS.

139. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

$2^{n-1}(2^n-1)$ is a multiply perfect number of multiplicity 2 when 2^n-1 is prime. Prove that there are no other multiply perfect numbers containing only 2 distinct primes.

140. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Determine (any way) whether the Diophantine equation $\left(\frac{2x-1}{3}\right)^3=x^2+y^2$ has any positive integer solutions.

141. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Given that the highest factor of a prime p contained in $m!$ is p^{m-s}; find general expressions involving p and m and s, from which, when a solution is possible, m can be determined when s is a given integer and p is a given prime. Is it then possible in any case to have more solutions than one?

AVERAGE AND PROBABILITY.

181. Proposed by W. J. GREENSTREET, M. A., Editor of the Mathematical Gazette, Stroud, England.

At a sea-side excursion for x men there are boats enough for q men and carriages enough for z. But p do not care for driving, and q would feel indifferently comfortable on the water, while the rest do not care either way. Each man has what he prefers as long as a seat is left for him in carriages or boats, and those who do not care either way choose at random. Find the chance that all will be satisfied.

182. Proposed by L. MORDELL, Philadelphia, Pa.

Out of n straight lines whose lengths are 1, 2, 3, 4, ..., n inches, respectively, the number of ways in which 4 may be chosen which will form a quadrilateral in which a circle may be inscribed is $\frac{1}{48}[2n(n-2)(2n-5)-3+3(-1)^n]$.

MISCELLANEOUS.

163. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

Two straight streams of different volumes and velocities come together. Find the path of a body floating in mid-current of either.

164. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Find the number of real roots of the equation $100\sin x = x$, and show the largest root is approximately 96.10. Find $\tan 39°$ to three places of decimals. How many real roots of $\tan x = 1/x^2$ lie between 0 and 2π?

165. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Prove that $\tan^{-1}\frac{n}{n+1} + \tan^{-1}\frac{1}{2n+1} = \frac{1}{4}\pi$.

166. Proposed by F. H. SAFFORD, Ph. D., The University of Pennsylvania.

Several equal rectangular boxes are placed in a row with uniform intervals between the boxes and a passageway along one side of the row. Find the least width of the passageway permitting a box to be removed from the row without moving adjacent boxes. This problem arose during the construction of a room for storage batteries.

167. Proposed by DR. OSWALD VEBLEN, Princeton University, Princeton, N. J.

If possible, arrange 43 objects, say the numbers 0, 1, 2, ..., in 43 sets of seven each such that every pair of objects lies in one and only one set of seven. It will then be true that two sets of seven have in common one and only one object.*

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*The analogy of this arrangement with the points in an ordinary projective plane is obvious, the "objects" corresponding to points and the sets of seven to lines. A similar arrangement with a smaller number of objects is the triple system,

0 1 2 3 4 5 6
1 2 3 4 5 6 0
3 4 5 6 0 1 2

in which seven objects are arranged in sets of three. Other examples are to be found in an article by O. Veblen and W. H. Bussey on *Finite Projective Geometries* in the Transactions of the American Mathematical Society, Vol. 7, page 241, and in an article by O. Veblen on *The Foundations of Geometry* in The Popular Science Monthly for January, 1906.

NOTES AND NEWS.

Professor M. W. Haskell, of the University of California, has been promoted to a full professorship.

Professor H. P. Manning, of Brown University, has been promoted to an associate professorship of mathematics.

At the University of Rochester, Dr. A. S. Gale has been promoted to the Fayerweather Professorship of Mathematics.

Professor J. J. Quinn has charge of Manual Training and Mechanical Drawing in the Scottdale (Pa.) Public Schools.

Professors P. A. Lambert and A. E. Meake, of Lehigh University, have been promoted to full professorships of mathematics.

Milton L. Comstock, who for forty years was a teacher in Knox College, died at his home in Galesburg, Illinois, November 8, 1906, at the age of 82. Professor Comstock was elected to the chair of mathematics in Knox College in 1858 and held the position until 1898, when he was made Professor Emeritus. For years he was prominent in Teachers' Institute work in Illinois. He was a subscriber to the MONTHLY for several years.

BOOKS AND PERIODICALS.

A Laboratory Course in Physics for Secondary Schools. By Robert Andrews Millikan, Ph. D., Assistant in Physics in The University of Chicago, and Henry Gordon Gale, Ph. D., Instructor in Physics in The University of Chicago. 8vo. Flexible Cloth Cover, x+134 pages. Boston and Chicago: Ginn & Co.

This little laboratory manual designed for secondary schools contains 51 experiments covering a wide range of topics. The description of the experiments is adequate and the illustrations are good. The book will be welcomed by teachers of physics whose appropriation for apparatus is very limited, for much of the apparatus described can be made at home.

B. F. F.

Algebra for Secondary Schools. By Webster Wells, S. B., Professor of Mathematics in the Massachusetts Institute of Technology. 8vo. Flexible Cloth Back, x+462 pages+50 pages of answers. Price, $1.20. Boston and Chicago: D. C. Heath & Co.

Among the distinctive features of this work are the use of formulas of physics and the introduction of problems involving the elementary laws of physics occurring at intervals throughout the book; graphical work occurring wherever the equation is introduced and given sufficient scope to acquaint the student with the great value of this work. Professor Wells has brought together in this book the best features of his other algebras.

B. F. F.

The Elements of Physics. By S. E. Coleman, A. B., A. M. (Harvard), Head of the Science Department and Teacher of Physics in the Oakland (Cal.) High School. 8vo. Cloth, 448 pages. Price, $1.25. Boston and Chicago: D. C. Heath & Co.

This book is well suited to High School use. The treatment of the general principles is ample, the selection of exercises is good, and the order of treatment of the various subjects is logical. B. F. F.

Text-Book in Algebra. By Webster Wells, S. B., Professor of Mathematics in the Massachusetts Institute of Technology. 8vo. Half Leather Back, xi+561 pages. Price, $1.50. Boston and Chicago: D. C. Heath & Co.

This text-book contains over 4000 examples and problems, and a full treatment of all subjects of ordinary algebra. Free use is made of graphs. The book contains an Index in which are given references to all important operations and definitions. B. F. F.

Analytical Geometry. A First Course. By W. H. Maltbie, Woman's College of Baltimore, Md. 8vo. Paper Back, 142 pages. Published by the Author.

The aim of the author in preparing this book is to give a clear exposition of the methods of Analytical Geometry, leaving the working out of theorems largely to the student. In addition to the general treatment of the straight line and the conics, five appendices A, B, C, D, E, are added, in which are brief discussions of Infinities of Various Orders, Functionality, Permissible Operations, Projections, and Imaginaries. B. F. F.

Inductive Plane Geometry with Numerous Exercises, Theorems, and Problems for Advanced Work. By G. Irving Hopkins, Instructor in Mathematics and Astronomy, High School, Manchester, N. H. Revised Edition. Half Leather, vi+208 pages. Price, 75 cents. Boston and Chicago: D. C. Heath & Co.

Original work is the main feature of the method used by the author. "The early introduction of triangles, the minimum use of the method of superposition, numerical problems involving the use of the metric system of measures, and problems showing the applications of algebra to geometry" are special features of this edition.

On page 29 is given a direct proof of the theorem, *If the bisectors of the base angles of a triangle are equal, the triangle is isosceles.* This demonstration was published in Vol. IX, p. 43, of the MONTHLY, and it is believed was the first direct proof of the theorem ever published. B. F. F.

Practical Business Arithmetic. By John H. Moore, Commercial Department of the Charleston High School, Boston, and George W. Miner, Commercial Department, Westfield (Mass.) High School. 8vo. Cloth, viii+449 pages. Price, $1.00.

Some of the points of superiority claimed for this book are: (1) the emphasizing of fundamental operations; (2) freedom from useless definitions and theory; (3) problems dealing with live matter and correlated with the business activities of to-day; (4) graphic representations, scales, plots, and estimates given their proper share of attention; (5) and the illustration of a variety of business forms in attractive script.

The book is printed on a good quality of paper, is neatly bound, is very attractive in general appearance, and will prove of great value to all teachers of commercial arithmetic. B. F. F.

Text-Book of Mechanics. By Louis A. Martin, Jr. (M. E., Stevens; A. M., Columbia), Assistant Professor of Mathematics and Mechanics in Stevens Institute of Technology. Vol. I, Statics. First Edition. xii+142 pp. 8vo. Cloth. Price, $1.25. New York: John Wiley & Sons.

The author states in his preface that the work is based upon notes prepared for the use of Freshman and Sophomore classes at Stevens Institute of Technology. Analytical Geometry is used but no use is made of the Calculus. Numerous examples are introduced to illustrate each principle. B. F. F.

The Development of Symbolic Logic. A Critical-Historical Study of the Logical Calculus. By A. T. Shearman, M. A. 8vo. Cloth, x+242 pages. Price, 5 shillings. London: Williams and Norgate.

This book aims to show that there has been made during the last fifty years a definite advance in Symbolic Logic. The growth of the subject from the time when Boole wrote his *Laws of Thought* to the time when Russel, following for the most part the lines laid down by Peano, showed how to deal with a wider range of problems than those considered by Boole, has been traced. The book will be of great value to logicians and students of pure mathematics. B. F. F.

A First Course in Physics. By Robert Andrews Millikan, Ph. D., Assistant Professor of Physics in The University of Chicago, and Henry Gordon Gale, Ph. D., Instructor in Physics in The University of Chicago. 12mo. Cloth, viii +488 pages. Boston and Chicago: Ginn & Co.

This book, which is intended for a third year High School pupil, has grown out of the author's experience in developing the work in Physics in the School of Education of The University of Chicago. It emphasizes the historical and practical phases of the subject and closely connects it with the experience and observation of every day life. Human interest is added to the book by incorporating excellent portraits and brief biographies of such noted physicists as Maxwell, Faraday, Helmholtz, etc. A first class book for the High School. B. F. F.

Text-book on the Strength of Materials. By S. E. Slocum, Assistant Professor of Mathematics in the University of Illinois, and E. L. Hancock, Instructor in Applied Mechanics in Purdue University. 8vo. Cloth, xii+314 pages. Illustrated. List price, $2.00.

This book was produced with the aim of thoroughly representing the best theory and practice in an elementary form for the use of third year students in technical and engineering schools. It is divided into two parts; the first part containing 216 pages, presents the theoretical side of the subject, and the second part the experimental side. The entire subject is systematically developed, and in the treatment rigor and simplicity are happily combined. The book is very handsomely gotten up, the typography, illustrations, and mechanical execution being very good. B. F. F.

Essentials of Geometry, Plane and Solid. By Webster Wells. Half Leather, vii+402 pages. Price, $1.25. Boston and Chicago: D. C. Heath & Co.

In this book, the author recognizes the needs of the student, and meets them in such a way as to arouse his interest. The exercises, about 800, are carefully selected, the definitions are accurate, the demonstrations rigorous, and the discussions clear and logical. B. F. F.

Space and Geometry in the Light of Physiological, Psychological, and Physical Inquiry. By Dr. Ernest Mach, Emeritus Professor in the University of Vienna. From the German by Thomas J. McCormack, Principal of the La Salle-Peru Township High School. 8vo. Cloth, 148 pages. Price, $1.00. Chicago: The Open Court Publishing Co.

This little book is made up of three essays which were originally written for the *Monist* and which have appeared in that journal. The first essay treats "On Physiological, as Distinguished from Geometrical, Space"; the second, "On the Psychological and Natural Development of Geometry"; and the third, "Space and Geometry from the Point of View of Physical Inquiry." Dr. Mach's writings on a wide range of scientific subjects are well and favorably known to all readers interested in the progress of science. His familiarity with physiological, psychological, and physical theories has enabled him to discuss the subject from different points of view, and to throw light on the many divergent forms which the science of space has historically assumed. His discussion is thoroughly interesting and instructive. B. F. F.

Philolaus. By William Romaine Newbold, University of Pennsylvania. Reprinted from Archiv für Geschichte der Philosophie, XIX Band. Heft 2, 1905. Pamphlet, 176—217 pages.

The author attempts, in this article, to give an interpretation of several passages which were not satisfactorily explained by Boeckh in his monograph upon Philolaus, nor by any of his successors. The first is Philolaus' meaning of "Embodying" and "Splitting" ratios, a function which he applies to number. The second is the nature of the principles assumed by Philolaus and termed by him περαίνοντα and ἄπειρα. In the course of the author's discussion, he brings out a number of historical facts which will be of considerable interest to writers on the history of mathematics and astronomy.

A few typographical errors, and one error of statement on page 208, patent to the geometrician, exist. Here the author says, "Such a polygon [a polygon of fifty-eight sides] cannot even to-day be inscribed in a circle by any means known to mathematics." He, of course, means a regular polygon; and even then his statement is not true, for such a polygon is easily inscribed in a circle by means of a multisector, for example. By means of a straight edge and a pair of compasses, the problem is, of course, impossible. B. F. F.

Groups Generated by Two Operators which Transform Each Other into the Same Power. By Dr. G. A. Miller. Obilka z Prac Matematy czno-fizycznyck, T. XVII. Pamphlet, 4 pages.

The Value of Science. By M. H. Poincare. Translated by Dr. George Bruce Halsted. Reprinted from Popular Science Monthly. Pamphlet, 193—206 pages.

The Cattle Problem of Archimedes. By Professor Mansfield Merriman, Lehigh University. Reprinted from Popular Science Monthly. Pamphlet, 660—665 pages.

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ON LINEAR ALGEBRAS.

By DR. L. E. DICKSON.

In the theory of algebraic numbers we consider such systems of numbers as the set of all numbers $r+si+ti^2$, in which r, s, t are rational, while i is an irrational number satisfying an equation $x^3-\beta x-b=0$ with rational coefficients. This set of numbers is called a field (or domain) since the sum, difference, product, or quotient, of any two of the numbers is likewise an unique number of the set. If we put $i^2=j$, we may consider this field to be a linear algebra composed of all numbers $r+si+tj$ (r, s, t rational), in which the "units" $1, i, j$ satisfy the relations

$$1^2=1,\ 1.i=i.1=i,\ 1.j=j.1=j, \tag{1}$$
$$i^2=j,\ i.j=j.i=b+\beta i,\ j^2=bi+\beta j. \tag{2}$$

In general, a linear triple algebra is determined by the "multiplication table" of its units [(1) and (2) in the above case] and by the field over which the coördinates r, s, t range [the field of rational numbers in the above case]. There is a very extensive literature on linear *associative* algebras, in which $i(jk)=(ij)k$ for any three units (not necessarily distinct). But only in rare instances, such as for quaternions, is division always possible; while then multiplication is not commutative. Another class of highly interesting linear algebras has been considered recently.* In these, multiplication is commutative and distributive, but not always associative, while *division is always uniquely possible*. We shall here

*Dickson, *Goettingen Nachrichten*, 1905, pp. 358—393; *Transactions American Mathematical Society*, Vol. 7 (1906), pp. 370—390; 514—522.